- Relation Between Second order linear ODE & 2-varible ODEs.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix}$$

$x' = ax + by + e.$
$\Updownarrow$
$by = x' - ax - e.$

$$\rightsquigarrow x'' = ax' + by' + e$$
$$= ax' + b(cx + dy + f) + e$$
$$= ax' + bcx + d\,\underline{by} + bf + e.$$
$$= \underline{ax'} + bcx + d(x' - ax - e) + bf + e.$$
$$= (a+d)x' + (bc - da)x - de + bf + e.$$
$$\rightsquigarrow \quad x'' + Ax' + Bx + C = 0.$$

Second order linear ODE. $\rightsquigarrow$ Consider homogeneous case.

$$\underline{x'' + Ax' + Bx = 0.} \quad (*)$$

How to solve? Consider $\lambda^2 + A\lambda + B = 0.$ $\rightsquigarrow$ solution λ_1, λ_2.

Try $e^{\lambda t}$: $\lambda^2 e^{\lambda t} + \lambda A e^{\lambda t} + B e^{\lambda t} = 0$.

i.e. $(\lambda^2 + A\lambda + B)e^{\lambda t} = 0.$

$\Rightarrow e^{\lambda t}$ is a solution of $(*)$ iff $\lambda^2 + A\lambda + B = 0.$ ☆.

Conclusion: ① λ_1, λ_2 : 2 different real roots.

$$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}.$$

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$$x(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t}.$$

Substitute $x = te^{\lambda t}$.

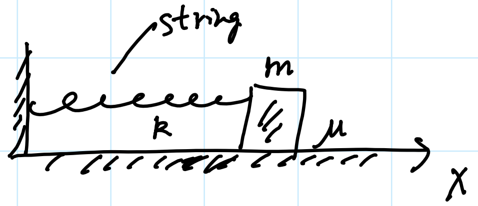
$$\text{equ.} \Rightarrow \Big(\underbrace{(\lambda^2 + A\lambda + B)}_{=0} t + \underbrace{2\lambda + A}_{=0} \Big) e^{\lambda t} = 0. \quad \checkmark$$

③ $\lambda_1 = a + bi$, $\lambda_2 = a - bi$.

$$\longrightarrow x(t) = c_1 \underline{e^{at}} e^{ibt} + c_2 \underline{e^{at}} e^{-ibt}$$

$$= e^{at} \big(c_1 \sin(bt) + c_2 \cos(bt) \big).$$

Example: Harmonic motion.



$x(0) = x_0$, $\quad x'(0) = v_0$.

$m x'' = -kx - \mu x'$. $\quad k > 0, \mu > 0$.

$$\Rightarrow x'' + \frac{\mu}{m} x' + \frac{k}{m} x = 0.$$

Consider $\lambda^2 + \frac{\mu}{m}\lambda + \frac{k}{m} = 0 \Rightarrow \lambda = \dfrac{-\frac{\mu}{m} \pm \sqrt{(\frac{\mu}{m})^2 - 4\frac{k}{m}}}{2}$.

For simplicity. $\mu = 0$. $\rightsquigarrow \lambda = \pm i \sqrt{\frac{k}{m}}$.

$$\rightsquigarrow x = c_1 \sin\Big(\sqrt{\tfrac{k}{m}}\, t\Big) + c_2 \cos\Big(\sqrt{\tfrac{k}{m}}\, t\Big).$$

c_1, c_2 is determined by x_0, v_0. $\checkmark$

Inhomogeneous case: $\underline{x'' + Ax' + Bx = f(t).}$ $\quad (**)$

Step 1: Solve homogeneous case: $x'' + Ax' + Bx = 0$. $\quad (*)$

Step 2: find a special solution: $\tilde{x}(t)$ of $(**)$.

Step 3: $x(t) = \boxed{} + \tilde{x}(t)$

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$\hookrightarrow$ solutions of (*).